NORTH-HOLLAND

Regular Hankel Matrices Over Integral Domains*

M. C. Gouveia
Dep. de Matemática
Univ. de Coimbra
3000 Coimbra, Portugal

Submitted by Graciano de Oliveira

ABSTRACT

In an earlier paper the author proved that a finite Hankel matrix over a field has a rank factorization given by the extended Frobenius theorem. This paper states under what conditions such results can be extended to integral domains. In such cases a characterization of regular Hankel matrices is also given. © Elsevier Science Inc., 1997

1. INTRODUCTION

Let R be an integral domain, i.e., a commutative ring with multiplicative identity and with no zero divisors, and let $\text{Mat}(R)$ denote the category of finite matrices with elements in R.

Let $A \in \text{Mat}(R)$ be an $m \times n$ matrix. An $n \times m$ matrix G in $\text{Mat}(R)$ is called a (1)-*inverse* of A (also called *generalized inverse*) if G satisfies the equation

$$AGA = A. \tag{1}$$

A matrix A which has a (1)-inverse $G = A^{(1)}$ is said to be *regular*. If G satisfies (1) and

$$GAG = G, \tag{2}$$

* This research was partially supported by IT-Pólo Coimbra, Portugal.

0024-3795/97/$17.00
PII S0024-3795(96)00003-1

it is called a $(1,2)$-inverse of A. It is clear from the definition that if A is regular then A has a $(1,2)$-inverse.

An element of R is called a unit if it has an inverse in R.

We now introduce some notation. If A is an $m \times n$ matrix, let $\alpha = \{i_1, \dots, i_l\}$ be a subset of $\{1, \dots, m\}$, and let $\beta = \{j_1, \dots, j_k\}$ be a subset of $\{1, \dots, n\}$. We denote by $A_\beta^\alpha = A[i_1, \dots, i_l | j_1, \dots, j_k]$ the submatrix of A determined by the rows indexed by α and the columns indexed by β. If $\alpha = \beta$, then $A_\alpha^\alpha = A[i_1, \dots, i_l]$. If $\alpha = \{1, \dots, m\}$ or $\beta = \{1, \dots, n\}$, then A_β^α is simply denoted, respectively, by A_β or A^α.

For a square matrix $A = (a_{ij})$, the determinant of A is denoted by $|A|$ (or det A), and the cofactor of a_{ij} in A, i.e., the coefficient of a_{ij} in the expansion of $|A|$, by $\delta|A|/\delta a_{ij}$.

The (determinantal) rank of a matrix A is defined as the size of the largest nonvanishing minor of A, and denoted by rank A.

An $m \times n$ matrix of rank ρ is said to have a full rank factorization $A = BC$ if the $m \times \rho$ matrix B and the $\rho \times n$ matrix C are of rank ρ.

Let A be an $m \times n$ matrix, and let $\rho \leqslant \min\{m, n\}$. By $C_\rho(A)$ we mean the *ρth compound matrix* of A, i.e., the matrix of Mat(R) whose (α, β) entry is $|A_\beta^\alpha|$, where α and β run over all ρ-element subsets respectively of $\{1, \dots, m\}$, and $\{1, \dots, n\}$, and where in any row or column the order of precedence is decided by the order of magnitude of the numbers whose digits are the column or row numbers of the minors.

Some properties are (see [5]):

$$[C_\rho(A)]^T = C_\rho(A^T); \tag{1.1a}$$

by the Cauchy-Binet formula,

$$C_\rho(AB) = C_\rho(A)C_\rho(B); \tag{1.1b}$$

$$\operatorname{rank} C_\rho(A) = 1. \tag{1.1c}$$

2. HANKEL MATRICES OVER INTEGRAL DOMAINS

Let $H = (h_{i+j})$, $h_{i+j} \in R$, $i = 0, \dots, m-1$, $j = 0, \dots, n-1$, be an $m \times n$ Hankel matrix. We will assume $m \leqslant n$ unless indicated otherwise, and occasionally we denote H by $H_{m,n}$. If $m = n$ we will write H_n.

Let $D_{\mu-1} := |H_\alpha^\alpha|$, $\alpha = \{1, \dots, \mu\}$ if $\mu = 1, \dots, m$, and let $D_{\mu-1} := 1$ if $\mu = 0$. The maximal natural number μ $(0 \leqslant \mu \leqslant m)$ such that $D_{\mu-1}$ is different from zero is, by definition, the *r-characteristic* of $H_{m,n}$ [denoted by $r(H)$].

Now we mention some properties of Hankel matrices which are consequences of their special structure.

Let $H_{m,n}$ be such that $r(H) = r$. Then $D_r = 0$ and, from Laplace's expansion of D_r with respect to the $(r+1)$th row, we have

$$\begin{aligned} D_r &= \sum_{j=0}^{r} h_{r+j} \frac{\delta D_r}{\delta h_{r+j}} \\ &= 0. \end{aligned}$$

Since $\delta D_r / \delta h_{2r} = D_{r-1}$, from this equality it follows that

$$D_{r-1} h_{2r} = -\sum_{j=0}^{r-1} h_{r+j} \frac{\delta D_r}{\delta h_{r+j}}.$$

Let $\alpha_j \in R$ be such that $\alpha_j = -\delta D_r / \delta h_{r+j}$, $j = 0, \dots, r$. Since $D_{r-1} \neq 0$, over the field of quotients we have

$$h_{2r} = \sum_{j=0}^{r-1} \beta_j h_{r+j}, \tag{2.1}$$

where $\beta_j = \alpha_j / \alpha_r$, $j = 0, \dots, r-1$. Moreover, since $D_r = 0$ and $D_{r-1} \neq 0$, the $(r+1)$th column of D_r is linearly dependent on the first r columns. So, taking into account (2.1) and the Hankel structure of $H[1, \dots, m \mid 1, \dots, r+1]$, we have

$$h_\mu = \sum_{j=0}^{r-1} \beta_j h_{\mu-j-1} \tag{2.2}$$

for $\mu = r, \dots, r+m-1$.

It is clear that the equalities (2.1) and (2.2) are always true over R if D_{r-1} is a unit of R. In that case the R-module generated by the set of the first r columns of $H_{m,n}$ is free, and if $r(H) = \operatorname{rank} H$, each of the remaining $n - r$ columns is a linear combination of these. So, since all the elements on each column of $H_{m,n}$, except the last, stand also in the previous column with a shift toward the top of one position, we can extend (2.2) to $\mu = m + r, \dots, m + n - 2$.

Let $H = (h_{i+j})$ be a Hankel matrix over R such that rank $H = r(H) = 1$. Then h_0 is nonzero, and from (2.2) follows a complete characterization of H. In fact, there is α_0 in R such that

$$h_0^{\mu} h_{\mu} = \alpha_0^{\mu} h_0, \tag{2.3}$$

or, if h_0 is a unit of R, such that

$$h_{\mu} = \alpha_0^{\mu} h_0^{1-\mu}$$

for $\mu = i + j = 0, \dots, m + n - 2$.

We notice that if $r(H) = 1$ and $m + n - 2 \neq 0$, then from the definition it follows that $\alpha_0 = h_1$. Consequently, if $h_1 = 0$ then $h_{i+j} = 0$ for $i + j \neq 0$.

In the case where $r(H) > 1$, let

$$D_{r-1}^{(k)} := |H_{\beta}^{\beta}|, \quad \beta = \{1 + k, \dots, r + k\}, \qquad k = 1, \dots, m - r,$$

$$D_{r-1}^{(0)} := D_{r-1}.$$

From (2.2) the following recurrence formula follows:

$$D_{r-1}^{(k)} D_{r-1} = D_{r-1}^{(k-1)} D_{r-1}^{(1)},$$

$k = 1, \dots, m - r$. If D_{r-1} is a unit of R, then this formula is equivalent to

$$D_{r-1}^{(k)} = \left(D_{r-1}^{(1)}\right)^k \left(D_{r-1}\right)^{-k+1}.$$

If $r(H) = r$ and rank $H = \rho$, then the extended Frobenius theorem [2] provides the following full rank factorization over the field of fractions of R,

$$H_{m,n} =: \begin{pmatrix} I_r & O \\ X & O \\ O & I_k \end{pmatrix} \begin{pmatrix} H^{\alpha} \\ H^{w} \end{pmatrix} := \begin{pmatrix} H^{\alpha} \\ H^{\tau} \\ H^{w} \end{pmatrix} \tag{2.4}$$

with $\alpha = \{1, \dots, r\}$, $\tau = \{r+1, \dots, m-k\}$, $w = \{m-k+1, \dots, m\}$, $k := \rho - r$, and $X = H^{\tau}K$, where K denotes an $n \times r$ right inverse of H^{α}.

DEFINITION 1. If a Hankel matrix H admits a factorization in $\mathrm{Mat}(R)$ defined by the equality (2.4), then H is said to have an *EF-rank factorization* in $\mathrm{Mat}(R)$.

THEOREM 2. *If there is an EF-rank factorization in* $\mathrm{Mat}(R)$ *for a Hankel matrix* $H \in \mathrm{Mat}(R)$, *then it is unique.*

Proof. Suppose that there are two different EF-rank factorizations of H in $\mathrm{Mat}(R)$. This means that there are X, X' in $\mathrm{Mat}(R)$ such that $XH^{\alpha} = H^{\tau}$ and $X'H^{\alpha} = H^{\tau}$ with $X \neq X'$.

Since H^{α} is full row rank, then H^{α} is right invertible in the quotient field of R. Consequently, H^{α} is right cancellable in $\mathrm{Mat}(R)$. Hence, $XH^{\alpha} = X'H^{\alpha}$ implies $X = X'$. So we have a contradiction. ∎

The existence of the Frobenius rank factorization (2.4) for any Hankel matrix over the quotient field means that the rows of H^{τ} are a linear combination of the rows of H^{α}, where the coefficients are given by the matrix X. Since determinantal ranks over an integral domain and its quotient field coincide, then $\operatorname{rank}[(H^{\alpha})^T\ (H^{\tau})^T]^T = \operatorname{rank} H^{\alpha}$ in $\mathrm{Mat}(R)$. Therefore, we can extend the Frobenius theorem to matrices over R if and only if we can define X in $\mathrm{Mat}(R)$ or, which is the same, if and only if there are coefficients in R such that the rows of the matrix H^{τ} are a linear combination of the rows of H^{α}. This observation and Theorem 2 immediately lead to the following.

THEOREM 3. *Let* $H_{m,n}$ *be a Hankel matrix in* $\mathrm{Mat}(R)$ *such that* $\operatorname{rank} H = \rho$ *and* $r(H) = r$, $0 < r \leqslant \rho < m$. *Then* $H_{m,n}$ *has an EF-rank factorization in* $\mathrm{Mat}(R)$ *if and only if the set of the first* r *rows of* $H_{m,n}$ *is a basis for the R-module generated by the first* $m-k$ *rows of* $H_{m,n}$, *where* $k = \rho - r$.

From (2.4) and the Cauchy-Binet formula we have

$$C_{\rho}(H) = (|E^{\alpha}|)_{\binom{m}{\rho}\times 1}(|F_{\beta}|)_{1\times\binom{n}{\rho}}. \tag{2.5}$$

Hence, $C_\rho(H)$ is a matrix with at least $\binom{m-k}{\rho}$ null rows corresponding to $|H_\beta^\alpha|$, with α a ρ-subset of $\{1,\dots,m-k\}$, and with the remaining rows being proportional to the row of the minors $|H_\beta^\alpha|$ taken from the first r and the last k rows of H, i.e., $\alpha = \{1,\dots,r,m-k+1,\dots,m\}$, considering β as a ρ-subset of $\{1,\dots,n\}$.

It is easy to check that if rank $H \neq 1$, then $C_\rho(H)$ is not always a Hankel matrix. Since H is a symmetric matrix, by (1.1a) it is clear that $C_\rho(H)$ is also symmetric.

In the particular case of rank $H = r(H) = r$, the main diagonal elements of $C_r(H)$ are the minors $D_{r-1}^{(k)}$, $k = 0,\dots,m-r$.

3. GENERALIZED INVERTIBILITY

The theory of generalized inverses of matrices over integral domains is quite well developed in the literature. For the characterization of regular matrices in Mat(R) we refer to the following theorem [7, Theorem 8]:

THEOREM 4. *Let A be an $m \times n$ matrix with rank ρ. Then the following are equivalent:*

(i) *A is regular.*
(ii) *$C_\rho(A)$ is regular.*
(iii) *A linear combination of all the $\rho \times \rho$ minors of A is equal to one.*

Also, generalized invertibility of Hankel matrices over fields and over some special particular rings was stated recently in [2]. The starting point of the investigation was the proof of the existence of a full rank factorization for any Hankel matrix over a field given by the extended Frobenius theorem. However, the statement is not true for any Hankel matrix over an integral domain. For example,

$$H = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}$$

over the ring R generated by 1, x^2, y^2, and xy in $\mathbb{R}[x, y]$ is a Hankel matrix such that rank $H = r(H) = 1$, which does not have a rank factorization over R. Nevertheless, over the field of fractions,

$$\begin{pmatrix} 1 \\ [xy \quad y^2]K \end{pmatrix} \begin{pmatrix} x^2 & xy \end{pmatrix},$$

where K is a right inverse of $H^{\{1\}}$, is a full rank decomposition of H. However, over R there is no such K and neither $H^{\{1\}}$ nor H is regular.

Also in the case where an EF-rank factorization exists, the matrix is not always regular. For example, let $R = \mathrm{Z}$ and let

$$H = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

so that $r(H) = \operatorname{rank} H = 1$. In this case H has an EF-rank factorization in $\operatorname{Mat}(\mathrm{Z})$,

$$H = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \end{pmatrix},$$

but neither H nor $H^{\{1\}}$ is regular.

Some other similar examples lead us to conjecture that regularity of a Hankel matrix such that $r(H) = r$ is related to the regularity of the submatrix $H_{r,n} := H^{\alpha}$, $\alpha = \{1, \dots, r\}$.

As many problems become tractable when a regular matrix has a rank factorization, this also leads us to a characterization of regular Hankel matrices which have EF-rank factorization.

The main results will be proved in the last section.

As, by Theorem 4 and (1.1c), the regularity of Hankel matrices over integral domains is related to matrices of rank one, first we prove some particular results.

THEOREM 5. *If $H = (h_{ij})$ is an $m \times n$ Hankel matrix of rank 1, then H is regular in* $\operatorname{Mat}(R)$ *if and only if there is a linear combination of the generating elements of H which is equal to one.*

Proof. From Theorem 4 it follows that H is regular if and only if there are g_{ji} in R such that

$$\sum_{i,j} g_{ji} h_{ij} = 1, \qquad i = 0, \dots, m-1, \quad j = 0, \dots, n-1.$$

By the definition of H, $h_{i+j} = h_{ij}$, $i + j = 0, \dots, m + n - 2$, are the generating elements of H. Therefore, there are $d_\mu \in R$, $\mu = i + j$, such that

$$\sum_{\mu=0}^{m+n-2} d_\mu h_\mu = 1,$$

which means that H is regular iff a linear combination of the elements in the first row and in the last column of H is equal to one. ■

COROLLARY 6. *Let H be an $m \times n$ Hankel matrix over R such that* rank $H = 1$ *and $r(H) = 0$. Then H is regular iff h_{m+n-2} is a unit of R. In this case $H = (h_{ij})$ with $h_{ij} = 0$ if $i + j \neq m + n - 2$.*

Proof. If $r(H) = 0$, then, because of the definition of the r-characteristic, H must be of the form

$$\begin{pmatrix} 0 & \cdots & 0 & h_m & \cdots & h_{n-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & h_m & \cdots & h_{n-1} & \cdots & h_{m+n-2} \end{pmatrix}.$$

Since every 2×2 minor is zero and since there are no zero divisors in R, it is easy to check that if $m \leqslant k < m + n - 2$, then h_k must be zero. Consequently, by Theorem 5, H is regular iff h_{m+n-2} is a unit of R. ■

COROLLARY 7. *If $H_n = (h_{ij})$ is a square Hankel matrix with rank* 1, *then H_n is regular in* Mat(R) *if and only if a linear combination of the main diagonal elements of H_n is equal to one.*

Proof. From Theorem 5, H_n is regular in Mat(R) iff the row matrix $H = (h_0\ h_1\ \cdots\ h_{n-1}\ \cdots\ h_{2n-2})$ is regular. This means that there is a column matrix G in Mat(R) such that $HG = 1$, i.e., such that $\Sigma g_i h_i = 1$, $i = 0, \ldots, 2n - 2$. Therefore, $(HG)^2 = 1$ implies that

$$\sum_{i,j=0}^{2n-2} g_i g_j h_i h_j = 1.$$

Expanding the sum on the left side of the last equality and separating the indices i, j according to parity, we obtain

$$\sum_{0 \leqslant i \leqslant n-1} g_{2i}^2 h_{2i}^2 + \sum_{1 \leqslant i \leqslant n-1} c_{2i-1}^2 h_{2i-1}^2 + 2 \sum_{1 \leqslant i < j \leqslant n-1} c_{2i-1} c_{2j-1} h_{2i-1} h_{2j-1}$$
$$+ 2 \sum_{0 \leqslant i \leqslant n-1} \sum_{1 \leqslant j \leqslant n-1} g_{2i} g_{2j-1} h_{2i} h_{2j-1} + 2 \sum_{0 \leqslant i < j \leqslant n-1} c_{2i} c_{2j} h_{2i} h_{2j} = 1 \tag{3.1}$$

Since every 2×2 minor of H_n is equal to zero, then $h_{2i-1}^2 = h_{2i-2}h_{2i}$, $i = 1, \dots, n-1$, and $h_{2i-1}h_{2j-1} = h_{2i}h_{2j-2}$ for $i < j$, $i, j = 1, \dots, n-1$. So every term in the sum (3.1) is a linear combination of elements of the main diagonal of H_n. ■

REMARK. The last corollary can be easily extended to a nonsquare Hankel matrix $H_{m,n}$ of rank 1 such that $m, n \geqslant 2$ and $m + n - 2$ is even. With similar arguments to those of the foregoing proof, we conclude that *$H_{m,n}$ is regular in* Mat(R) *if and only if there is a linear combination of the elements h_{2i}, $0 \leqslant 2i \leqslant m + n - 2$, which is equal to one.*

If $m + n - 2$ is odd, a similar reasoning is valid for all terms in the sum (3.1) except for $g_{m+n-2}^2 h_{m+n-2}^2$. In this case, we can prove that *$H_{m,n}$ is regular in* Mat(R) *if and only if there is a linear combination of the elements h_{2i}, $2i = 0, 2, \dots, m + n - 1$, together with h_{m+n-2}, which is equal to one.*

As we have already noticed, if a Hankel matrix has an EF-rank factorization, then the ρth compound matrix is of special structure with the nonzero rows proportional to the row which corresponds to $C_\rho(F)$. Nevertheless, matrices of this form are not always regular, as for example

$$X = \begin{pmatrix} x & x \\ x^2 & x^2 \end{pmatrix} \in \mathrm{Mat}(\mathbb{R}[x]).$$

So we can give the following characterization.

THEOREM 8. *Let $L_i = \delta_i L_s$, $i = 1, \dots, m$, where L_i are the row vectors of an $m \times n$ matrix A, and $\delta_i \in R$ with $\delta_s := 1$, $1 \leqslant s \leqslant m$. Then the matrix A is regular if and only if L_s is regular.*

Proof. From the definition of A it follows that rank $A \leqslant 1$. The case of rank $A = 0$ being trivial, let rank $A = 1$. If A is regular, then there is a linear combination of the elements of A which is equal to one over R. So there are c_{ji} in R such that

$$\sum_{j=1}^{n} \sum_{i=1}^{m} c_{ji} a_{ij} = 1,$$

which is equivalent to

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} c_{ji}\delta_i\right)a_{sj} = 1.$$

Hence, by Theorem 4, $L_s = [a_{s1} \ \cdots \ a_{sn}]$ is regular.

The rest of the proof is obtained by retracing, since we let $c_{ji} = 0$ for $i \neq s$. ∎

THEOREM 9. *Let $H_{m,n}$ be a regular Hankel matrix in* Mat(R) *with rank ρ and r-characteristic r such that $0 < r \leqslant \rho < m$. Then H has an EF-rank factorization in* Mat(R) *if and only if $H_{r,n}$ is regular.*

Proof. Let $H_{r,n}$ be a regular matrix. Since $r(H_{m,n}) = r = \operatorname{rank} H_{r,n}$, $H_{r,n}$ is also of full row rank. By Theorem 1 in [7] there exists a matrix K in Mat(R) such that $H_{r,n}K = I_r$. Consequently, $X = H^{\tau}K$ is a solution for the matrix equation $XH_{r,n} = H^{\tau}$ which belongs to Mat(R). Therefore (2.4) is a full rank factorization of H in Mat(R).

Suppose now that $H_{m,n}$ has an EF-rank factorization in Mat(R). Since H is regular, by Theorem 4 there are $c_{\alpha\beta}$ in R, where α, β run over the ρ-subsets of, respectively, $\{1,\dots,m\}$ and $\{1,\dots,n\}$, such that

$$\sum_{\alpha,\beta} c_{\alpha\beta}|H_{\beta}^{\alpha}| = 1.$$

Thus, by (2.4) and the Cauchy-Binet formula,

$$\sum_{\alpha,\beta} c_{\alpha\beta}|E^{\alpha}|\,|F_{\beta}| = 1,$$

which is equivalent to

$$\sum_{\beta} d_{\beta}|F^{\beta}| = 1$$

for some elements d_β in R. This means that F is right invertible in Mat(R). So there is an $n \times \rho$ matrix D in Mat(R) such that $FD = I_{r+k}$ and, consequently, such that $H_{r,n}D[1,\dots,n \mid 1,\dots,r] = I_r$. Hence $H_{r,n}$ is right invertible, and, as it is full rank, by Theorem 1 in [7], $H_{r,n}$ is also regular in Mat(R). ∎

Let $\tilde{D} = \det H[1,\dots,r,m-k+1,\dots,m|1,\dots,r,s-k+1,\dots,s]$, the nonzero minor given by Theorem 2.1 in [2]. Therefore, from Theorem 4, THEOREM 10. *Let $H \in \mathrm{Mat}(R)$ be an $m \times n$ Hankel matrix with r-characteristic r and rank ρ, $0 < r \leqslant \rho < m$. If $\tilde{D}$ is a unit of R, then H is a regular matrix with an EF-rank factorization.*

Proof. Considering the generalized Laplace expansion of $\tilde{D}$ with respect to the last k rows, we have

$$\tilde{D} = \sum_{\alpha, \beta} d_{\beta, \alpha} |H_\beta^\alpha|$$

with $\alpha = \{1, \dots, r\}$, β an r-subset of $\{1, \dots, r, s-k+1, \dots, s\}$, and $d_{\beta\alpha}$ the complementary minor of $|H_\beta^\alpha|$.

If $\tilde{D}$ is a unit of R, then there is $\Phi \in R$ such that

$$\sum_{\alpha, \beta} \Phi d_{\beta\alpha} |H_\beta^\alpha| = 1.$$

Hence, $C_r(H_{r,n})$ is regular. Consequently, $H_{r,n}$ is regular, and from Theorem 9 it follows that there is an EF-rank factorization of H in $\mathrm{Mat}(R)$. ∎

4. MAIN THEOREM

THEOREM 11. *Let $H \in \mathrm{Mat}(R)$ be an $m \times n$ Hankel matrix such that $r(H) = r$ and* rank $H = \rho$, $0 \leqslant r \leqslant \rho < m$. *Then H is a regular matrix with an EF-rank factorization if and only if $H[1, \dots, r, m-k+1, \dots, m|1, \dots, n]$ is regular.*

Proof.

(1) Let $0 < r = \rho$, and let H be partitioned as

$$H = \begin{pmatrix} H^\alpha \\ H^\sigma \end{pmatrix}$$

for $\alpha = \{1, \dots, r\}$ and $\sigma = \{r+1, \dots, m\}$. If $H[1, \dots, r|1, \dots, n]$ is regular, there is $\sum_{\alpha, \beta} c_{\alpha\beta} |H_\beta^\alpha| = 1$, $c_{\alpha\beta} \in R$, where $\alpha = \{1, \dots, r\}$ and β runs over all r-element subsets of $\{1, \dots, n\}$. Let $d_{\theta\mu} = c_{\alpha\beta}$ if $\theta = \alpha$ and $\mu = \beta$, and let $d_{\theta\mu} = 0$ otherwise. Since $|H_\mu^\theta| = |H_\beta^\alpha|$ for $\theta = \{1, \dots, r\}$ and $\mu = \{1, \dots, n\}$, then

$$\sum_{\theta, \mu} d_{\theta\mu} |H_\mu^\theta| = 1$$

when θ runs over all r-element subsets of $\{1,\dots,m\}$ and μ runs over all r-element subsets of $\{1,\dots,n\}$. So H is regular, and, since $H_{r,n}$ is regular, by Theorem 9 H has an EF-rank factorization. Suppose now that H is regular with an EF-rank factorization. Then $C_r(H)$ is regular in Mat(R), and from (2.5) it follows that the nonnull rows of $C_r(H)$ are proportional to the row L_1 with entries $|H_\alpha^\beta|$, $\alpha=\{1,\dots,r\}$, β an r-subset of $\{1,\dots,n\}$. Hence, by Theorem 8, $C_r(H_{r,n})$ is regular.

(2) Let $r<\rho$. From the proof of Theorem 9 we know that if H is regular and it has an EF-rank factorization, then F is right invertible. Since F is full rank, this means that $H[1,\dots,r,m-k+1,\dots,m|1,\dots,n]$ is regular in Mat(R). Moreover, regularity of $H[1,\dots,r,m-k+1,\dots,m|1,\dots,n]$ implies regularity of H, and since H^α and H^w are full rank and since every $\rho\times\rho$ minor of H is a linear combination of $r\times r$ minors of H^α and $(\rho-r)\times(\rho-r)$ minors of H_i^w, then H^α and H^w are also regular. Hence, by Theorem 9, H has an EF-rank factorization.

(3) Let $r=0$. Then from the Hankel structure of H it follows that

$$H:=\begin{pmatrix} O \\ I_k \end{pmatrix}(H^w)$$

is an EF-rank factorization of H, where $k=\rho$, $w=\{m-k+1,\dots,m\}$, and H^w is a lower Hankel matrix with full row rank. So H is regular if and only if H^w is regular. ∎

We observe that this proof of the first case of Theorem 11 provides a (1)-inverse of H given by $G=(H_{r,n}^{(1)}\ 0)$, whose (i,j)th entry is $g_{ij}=\sum_{\alpha,\beta}c_{\alpha,\beta}(\partial/\partial h_{ji})|H_\beta^\alpha|$.

From the structure of G it is clear that, in this particular case, H has a Hankel generalized inverse iff $H[1,\dots,r]$ has an upper triangular Hankel (1)-inverse.

Finally we remark how important the structure of a Hankel matrix is. In fact, it is known from [6] that a regular matrix over an integral domain R has a rank factorization iff every finitely generated projective module over R is free. But, if such regular matrix is Hankel, from Theorem 11 it follows that

THEOREM 12. *Every regular Hankel matrix over an integral domain has an EF-rank factorization if and only if the module generated by rows of F is free.*

The author is very grateful to the referee for his constructive criticism and his interesting suggestions.

REFERENCES

1 R. B. Bapat, Bhaskara Rao, and K. Manjunata Prasad, Generalized inverses over integral domains, *Linear Algebra Appl.* 140:181–196 (1990).

2 M. C. Gouveia, Generalized invertibility of Hankel and Toeplitz matrices, *Linear Algebra Appl.* 193:95–106 (1993).

3 I. S. Iohvidov, *Hankel and Toeplitz Matrices and Forms, Algebraic Theory* Birkhäuser, Boston, 1982.

4 N. Jacobson, *Lectures in Abstract Algebra*, Vol. I, Van Nostrand, 1951.

5 T. Muir, *Treatise on the Theory of Determinants*, Dover, New York, 1960.

6 K. Manjunata Prasad and K. P. S. Bhaskara Rao, Integral Domains over Which Every Regular Matrix Has Rank Factorization, Tech. Report, Indian Statistical Inst., Bangalore.

7 K. P. S. Bhaskara Rao, On generalized inverses of matrices over integral domains, *Linear Algebra Appl.* 49:179–189 (1983).

Received 4 November 1994; final manuscript accepted 16 November 1995